4050 Hw 3 2024 (Star-question: 4)

1. Show that $\inf X \geq \inf Y$ whenever $X \subseteq Y (\subseteq \mathbb{R})$ and hence that $m^*(A) \uparrow$ (i.e. $m^*(A) \leq m^*(B)$ if $A \subseteq B (\subseteq \mathbb{R})$).

2. Let $\mathcal{A}$ be an algebra of subsets of X. Show that $\mathcal{A}$ is an σ-algebra if (and only if) $\mathcal{A}$ is stable w.r.t countable disjoint unions:
$$\bigcup_{n=1}^{\infty} A_n \quad \text{whenever } A_n \in \mathcal{A} \ \forall n \in \mathbb{N} \text{ and } A_m \cap A_n = \emptyset \ \forall m \neq n.$$

3. Suppose $[a,b] (\subseteq \mathbb{R})$ is covered by a finite family $\mathcal{C}$ of open intervals. Show that $b-a \leq$ sum of lengths of intervals in $\mathcal{C}$ (by MI to $n := \#(\mathcal{C})$, the number of elements of $\mathcal{C}$).

*4. (cf. Royden 3rd, P52, Q51). Upper/Lower Envelopes of $f: [a,b] \to \mathbb{R}$.
Define $h, g: [a,b] \to [-\infty, \infty]$ by, $\forall y \in [a,b]$,
$h(y) := \inf\{h_\delta(y) : \delta > 0\}$, where $h_\delta(y) := \sup\{f(x) : x \in [a,b], |x-y| < \delta\}$
$g(y) := \sup\{g_\delta(y) : \delta > 0\}$, where $g_\delta(y) := \inf\{f(x) : x \in [a,b], |x-y| < \delta\}$,
Prove the following:

a. $g \leq f \leq h$ pointwisely on $[a,b]$, and $\forall x \in [a,b]$,
$g(x) = f(x)$ iff f is lsc at x ($f(x) = h(x)$ iff f is usc at x)
so $g(x) = h(x)$ iff f is continuous at x.

b. If f is bounded (so g, h are real-valued) then g is lsc & h is usc

c. If φ is a lsc on $[a,b]$ such that $\varphi \leq f$ (pointwise) on $[a,b]$ then $\varphi \leq g$.
State and show the corresponding result for h.

d. Let $C_n := \{x \in [a,b] : h(x) - g(x) < \frac{1}{n}\}$ $\forall n \in \mathbb{N}$. Then $C := \bigcap_{n=1}^{\infty} C_n$ is exactly the set of all continuity points of f and is a G_δ-set.

Note. More suggestive notations for g, h are $\underline{f}$ and $\overline{f}$.

5. Let $f: [a,b] \to [m,M]$. For each $P \in Par[a,b]$, let $u(f;P)$ and $U(f;P)$ denote the lower/upper Riemann-sum-functions. Let $\{P_n : n \in \mathbb{N}\}$ be a sequence of partitions such that $P_n \subseteq P_{n+1}$ $\forall n$ and $\|P_n\| \to$ (the max of subinterval-lengths of P_n). Show that, $\forall x \in [a,b] \setminus A$
$\lim_n (u(f;P_n))(x) = \underline{f}(x)$ and $\lim_n (U(f;P_n))(x) = \overline{f}(x)$, where A denotes the union of all end-points of P_n $\forall n \in \mathbb{N}$